MATH 3341 — Fall 2019

Lab 06: LU Decomposition

Download `Math.3341.Lab.06.zip`, unzip it by following the Windows Instructions on WyoCourses. Change the current working directory of MATLAB to the unzipped folder, and type `edit lab_06_script` in the Command Window.

1 Solve a System with LU Decomposition

(a) Define matrix `A` and vector `b` as (1.1).

$$\underbrace{\begin{bmatrix} 7 & -26 & 45 & -47 \\ 1 & 2 & 3 & 4 \\ 2 & -11 & -12 & -13 \\ 4 & -17 & 30 & 35 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -98 \\ 30 \\ -108 \\ 200 \end{bmatrix}}_{\mathbf{b}} \tag{1.1}$$

(b) Calculate the LU decomposition `L`, `U` of the matrix `A`.

(c) Solve the following system (1.2) and store the solution to `z`.

$$L\mathbf{z} = \mathbf{b}. \tag{1.2}$$

(d) Then solve the following system (1.3) and store the solution to `x`.

$$U\mathbf{x} = \mathbf{z}. \tag{1.3}$$

(e) Check your solution by calculating the norm of the residual $\|A\mathbf{x} - \mathbf{b}\|_2$ and store the result to `res`.

2 Varying the Vector b

Suppose we want to solve the system for each integer value of m in between $m = 0$ and $m = 20$. This time use the LU decomposition of the system matrix; perform the decomposition only once and use the lower and upper triangular factors repeatedly to find each successive solution. Then generate a table (Table 1) and a plot (Figure 1) of the solution versus the integer m.

$$\begin{cases} 3x + y + z = m \\ x - 5y + 2z = 5 \\ 2x + y + 5z = 10 \end{cases} \tag{2.1}$$

To do this you'll follow the steps below:

(a) Define coefficient matrix `A` given in (2.1), and get the LU decomposition `L`, `U` of the matrix `A`.

(b) Define a vector `m` which ranges from 0 to 20 with step size 1.

(c) Then create a for-loop, of which the loop iterator `i` starts from `1` to `length(m)`. In the body of the loop, define a column vector `b` as the right-hand side of (2.1), where m should be the `i`th component of `m`. Then repeat (c) and (d) in Part 1. Store the solution `x` to the `i`th row of `X`.

(d) Format the output of `m` and `X` to a file called `solution.tex` as you did in Part 3 of Lab 05:

(i) Use `fprintf` to print out the setup for the *table* and *tabular* environments. The first column of the table is centered while the rest three columns are right-justified in LaTeX.

(ii) Between `\toprule` and `\midrule`, use `fprintf` to print out the heading of the table. The column widths are $4, 11, 11, 11$, respectively.

(iii) Between `\midrule` and `\bottomrule`, use a for-loop to print each row of the table. Note that the ith row of the table consists of the `i`th component of `m` and the `i`th row of matrix `X`. The column widths are $2, 9, 9, 9$, respectively. For floating point numbers, output 6 digits after the decimal point.

(iv) Call `type('solution.tex')` to print the content of `solution.tex`.

(e) Plot the solution versus m using a for-loop as you did in Part 4 of Lab 05:

(i) Get the size of `X` and assign it to `X_size`. Define a cell array `styles`, of which the entries are dashed line with hexagram, dotted line with pentagram, solid line with diamond.

(ii) The use a for-loop to plot each column of `X` versus `m` in the same figure window with the above styles.

(iii) Add labels, title, grid, legend as shown in Figure 1.

(iv) Save the plot to a file named `lab_06_plot.pdf`.

Type `diary('lab_06_output.txt')` in the Command Window, run the script file `lab_06_script.m`, and type `diary off` in the Command Window. Upload `lab_06_output.txt`, `lab_06_script.m`, `solution.tex`, and `lab_06_plot.pdf` to the folder `src` on Overleaf.

On Overleaf, open `body.tex` under the folder `LaTeX`. In the last section of the report, you will reproduce Section 3 using LaTeX. You may find the following helpful:

- You may use enviroments such as `equation`, `cases`, `figure`, and `table`.
- You may use `\textbf`, `\textsc`, `\textit`, `\textsl`, `\textsf`, `\emph`, `\bfseries`, `\itshape`, `\scshape` for formatting plain text.
- You may use `\includegraphics[width=amount unit]{/path/to/figure.pdf}` to specify the width of a figure. In our case, the width of the figure is `0.85\textwidth`.
- You may use `\ref{labelName}` to refer to figures, tables; use `\eqref{labelName}` to refer to equations.
- For special symbols, you may look them up in LaTeX.Mathematics.Symbols.pdf.
- You may use `\input{/path/to/solution.tex}` to include the table you got from MATLAB.

Recompile and submit the PDF file generated by Overleaf to WyoCourses.

3 Basics of LaTeX

3.1 LU Decomposition

Given the linear system (3.1)

$$\begin{cases} 3x + y + z = m \\ x - 5y + 2z = 5 \\ 2x + y + 5z = 10 \end{cases} \tag{3.1}$$

where $m = 0, 1, 2, \dots, 20$. Using LU Decomposition we can obtain the solution to the linear system (3.1) for corresponding m (see Table 1 and Figure 1).

Table 1: Solution to the linear system

m	x	y	z
0	−0.704225	−0.211268	2.323944
1	−0.323944	−0.197183	2.169014
2	0.056338	−0.183099	2.014085
3	0.436620	−0.169014	1.859155
4	0.816901	−0.154930	1.704225
5	1.197183	−0.140845	1.549296
6	1.577465	−0.126761	1.394366
7	1.957746	−0.112676	1.239437
8	2.338028	−0.098592	1.084507
9	2.718310	−0.084507	0.929577
10	3.098592	−0.070423	0.774648
11	3.478873	−0.056338	0.619718
12	3.859155	−0.042254	0.464789
13	4.239437	−0.028169	0.309859
14	4.619718	−0.014085	0.154930
15	5.000000	−0.000000	0.000000
16	5.380282	0.014085	−0.154930
17	5.760563	0.028169	−0.309859
18	6.140845	0.042254	−0.464789
19	6.521127	0.056338	−0.619718
20	6.901408	0.070423	−0.774648

3.2 Goldbach's Conjecture

Pursuing this type of analysis more carefully, Hardy and Littlewood in 1923 conjectured (as part of their famous *Hardy–Littlewood prime tuple conjecture*) that for any fixed $c \geq 2$, the number of representations of a large integer n as the sum of c primes $n = p_1 + \cdots + p_c$ with $p_1 \leq \cdots \leq p_c$ should be asymptotically equal to

$$\left(\prod_p \frac{p\gamma_{c,p}(n)}{(p-1)^c}\right) \int_{2 \leq x_1 \leq \cdots \leq x_c : x_1 + \cdots + x_c = n} \frac{dx_1 \cdots dx_{c-1}}{\ln x_1 \cdots \ln x_c}, \tag{3.2}$$

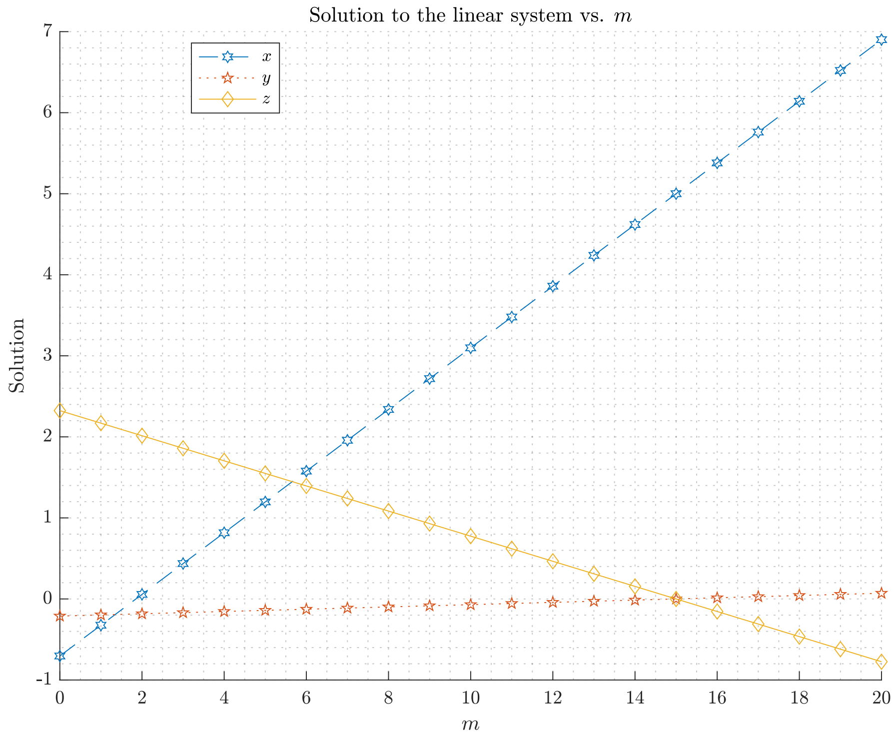

Figure 1: Solution to the linear system vs. m

where the product is over all primes p, and $\gamma_{c,p}(n)$ is the number of solutions to the equation $n = q_1 + \cdots + q_c \mod p$ in modular arithmetic, subject to the constraints $q_1, \ldots, q_c \neq 0 \mod p$. This formula (3.2) has been rigorously proven to be asymptotically valid for $c \geq 3$ from the work of Vinogradov, but is till only a conjecture when $c = 2$. In the latter case, the above formula simplifies to 0 when n is odd, and to

$$2\Pi_2 \left(\prod_{p|n;p\geq 3} \frac{p-1}{p-2} \right) \int_2^n \frac{dx}{(\ln x)^2} \approx 2\Pi_2 \left(\prod_{p|n;p\geq 3} \frac{p-1}{p-2} \right) \frac{n}{(\ln n)^2},$$

when n is even, where Π_2 is Hardy-Littlewood's twin prime constant

$$\Pi_2 := \prod_{p\geq 3} \left(1 - \frac{1}{(p-1)^2} \right) = 0.6601618158\ldots$$

This sometimes known as the extended Goldbach conjecture.
Reference: Goldbach's conjecture.